An analogue of Borg's uniqueness theorem in the case of indecomposable boundary conditions

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Abstract

An uniqueness theorem for the inverse problem in the case of a second-order equation defined on the interval [0,1] when the boundary forms contain combinations of the values of functions at the points 0 and 1 is proved. The auxiliary eigenvalue problems in our theorem are chose in the same manner as in Borg's uniqueness theorem are not as in that of Sadovničiĭ's. So number of conditions in our theorem is less than that in Sadovničiĭ's.

In the case of indecomposable boundary conditions (i. e. Sturm-Liouville conditions) several methods for the solution of the inverse problem are worked out in [1]–[9]. Extensive bibliographies for this problem can be found in Levitan [7]. In [10] a new method was proposed for the solution of the inverse problem with indecomposable boundary conditions. In this method instead of trasformation operators are utilized mappings T_λ of the spaces of the solutions defined by matrices.

It was shown that to restore uniquely the function $q(x)$ and the boundary conditions by a set of the eigenvalues λ_k of eigenvalue problem itself, a sets of the eigenvalues $z_{k,\,a}$ of two auxiliary eigenvalue problems, the "weight" numbers $\alpha_{k,\,a}$ and resides of the certain functions.

Sadovničiĭ showed the "weight" numbers $\alpha_{k,\,a}$ and resides of the certain functions is necessary for the unique restoration of the function $q(x)$ and the boundary conditions.

We will show that if two auxiliary eigenvalue problems are chose in the same manner as in Borg's uniqueness theorem ([2]) are not as in that

of Sadovničiĭ's ([10]), then the "weight" numbers $\alpha_{k,\,a}$ and resides of the certain functions are unnecessary for the uniqueness theorem.

1. We will consider the following problem:

$$\begin{aligned} ly &= -y'' + q(x)\,y = \lambda\,y, && (1)\\ U_1(y) &= y'(0) + a_{11}\,y(0) + a_{12}\,y(1) = 0, && (2)\\ U_2(y) &= y'(1) + a_{21}\,y(0) + a_{22}\,y(1) = 0 && (3) \end{aligned}$$

($a_{ij},\ i,\ j = 1,\ 2$ are real constants, $q(x) \in C^1[0,\ 1].$).

The problem defined in (1)–(3) will be called problem L. We denote by $\widetilde{L}$ a problem of type L but with different coefficients in the equation and with different parameters in the boundary forms. In all that follows, if certain symbol denotes a term from problem L, then the symbol$\,\widetilde{}\,$denotes the analogous term from problem $\widetilde{L}$. Everywhere the integral index k varies from 0 to ∞.

Along with problem L we consider two problems with decomposable boundary conditions: problems

L_1 :

$$\begin{aligned} ly &= -y'' + q(x)\,y = \lambda\,y,\\ U_{1,1}(y) &= y'(0) + a_{11}\,y(0) = 0,\\ U_{2,1}(y) &= y'(1) + a_{22}\,y(1) = 0 \end{aligned}$$

and L_2 :

$$\begin{aligned} ly &= -y'' + q(x)\,y = \lambda\,y,\\ U_{1,2}(y) &= y'(0) + a_{11}\,y(0) = 0,\\ U_{2,2}(y) &= y'(1) + a_{12}\,y(1) = 0 \end{aligned}$$

Let $\lambda_{k,\,1},\ \lambda_{k,\,2}$ be eigenvalues of these problems.

Theorem 1 *Let* $\lambda_k = \widetilde{\lambda}_k,\ \lambda_{k,\,1} = \widetilde{\lambda}_{k,\,1},\ \lambda_{k,\,2} = \widetilde{\lambda}_{k,\,2},\ a_{12} \neq a_{22}$; *then the coefficients of the equations and the constants in the boundary conditions of the problems* L *and* $\widetilde{L}$ *coincide, i.e.* $q(x) = \widetilde{q}(x),\ a_{ij} = \widetilde{a}_{ij}\ i,\ j = 1,\ 2.$

Proof. Making use of Borg's uniqueness theorem ([2, 7]) to problems L_1 and L_2 we have:

$$q(x) = \widetilde{q}(x), \qquad a_{11} = \widetilde{a}_{11}, \qquad a_{12} = \widetilde{a}_{12}, \qquad a_{22} = \widetilde{a}_{22}. \qquad (4)$$

To prove the theorem we must show that $a_{21} = \widetilde{a}_{21}$. We shall do it.

Let $y_1(x,\, \lambda)$ and $y_2(x,\, \lambda)$ be linearly independent solutions of equation (1) satisfying

$$y_1(0,\, \lambda) = 1, \quad y_1'(0,\, \lambda) = 0, \quad y_2(0,\, \lambda) = 0, \quad y_2'(0,\, \lambda) = 1.$$

Then asymptotic formulae

$$\begin{aligned}
y_1(x,\, \lambda) &= \cos \lambda x + \frac{1}{\lambda}\, u(x)\, \sin \lambda x + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \\
y_2(x,\, \lambda) &= \frac{1}{\lambda}\, \sin \lambda x - \frac{1}{\lambda^2}\, u(x)\, \cos \lambda x + \mathcal{O}\left(\frac{1}{\lambda^3}\right), \\
y_1'(x,\, \lambda) &= \lambda\, \sin \lambda x + u(x)\, \cos \lambda x + \mathcal{O}\left(\frac{1}{\lambda}\right), \\
y_2'(x,\, \lambda) &= \cos \lambda x + \frac{1}{\lambda}\, u(x)\, \sin \lambda x + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \\
\text{where } u(x) &= \frac{1}{2} \int_0^x q(t)\, dt
\end{aligned}$$

are true for $\lambda \in \mathbb{R}$ and λ sufficiently large ([11]).

It follows according to the condition of the theorem, that the eigenvalues of spectral problems L and $\widetilde{L}$ coincide. The eigenvalues of the problem L are the roots of the following entire function of the first order

$$\Delta(\lambda) = \begin{vmatrix} U_1(y_1(x, \lambda)) & U_1(y_2(x, \lambda)) \\ U_2(y_1(x, \lambda)) & U_2(y_2(x, \lambda)) \end{vmatrix}$$

([11]).

If we substitute the asymptotic formulae of solutions $y_1(x, \lambda)$, $y_2(x, \lambda))$ for the preceding equation, we obtain

$$\begin{aligned}
\Delta(\lambda) \quad &= \quad (a_{11} + a_{12}\, y_1(1,\, \lambda)) \cdot (y_2'(1,\, \lambda) + a_{22}\, y_2(1,\, \lambda)) - \\
&\quad -(1 + a_{12}\, y_2(1,\, \lambda)) \cdot (y_1'(1,\, \lambda) + a_{21} + a_{22}\, y_1(1,\, \lambda)) = \\
&= a_{11}\, \cos \lambda + a_{12} + \lambda\, \sin \lambda - u_1(1)\, \cos^2 \lambda + a_{21} + a_{22}\, \cos \lambda + \mathcal{O}\left(\frac{1}{\lambda}\right).
\end{aligned}$$

It follows from Weierstrass theorem about an entire function representation by its roots that

$$\Delta(\lambda) \equiv e^{a\lambda + b} \widetilde{\Delta}(\lambda),$$

where $\widetilde{\Delta}(\lambda)$ is a characteristic determinant of the problem $\widetilde{L}$ and a, b are certain numbers.

From here

$$\begin{aligned}
&\widetilde{\Delta}(\lambda) - e^{a\lambda+b}\Delta(\lambda) \equiv \\
&\quad \equiv (a_{11} - \widetilde{a}_{11}\, e^{a\lambda+b})\cos\lambda + (a_{12} - \widetilde{a}_{12}\, e^{a\lambda+b}) + \\
&\qquad +(1 - e^{a\lambda+b})\,\lambda\sin\lambda - (1 - e^{a\lambda+b})\,u_1(1)\cos\lambda - \\
&\qquad -(a_{21} - \widetilde{a}_{21}\, e^{a\lambda+b}) - (a_{22} - \widetilde{a}_{22}\, e^{a\lambda+b})\cos\lambda + \\
&\qquad +(1 - e^{a\lambda+b})\,\mathcal{O}\left(\frac{1}{\lambda}\right) \equiv 0.
\end{aligned}$$

The functions 1, $\sin\lambda$, $\cos\lambda$, $\cos 2\lambda$, $\lambda\cdot\sin\lambda$, $\mathcal{O}\left(\frac{1}{\lambda}\right)$ are linear independent functions with respect to argument λ. (It is easily verified by the definition of functions linear independence.) Consequently $a = 0$, $b = 0$ and

$$(a_{12} - \widetilde{a}_{12}) + (a_{21} - \widetilde{a}_{21}) + (a_{22} - \widetilde{a}_{22})\cos\lambda + \mathcal{O}\left(\frac{1}{\lambda}\right) \equiv 0.$$

This together with (4) then gives $a_{21} = \widetilde{a}_{21}$. $\square$

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